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AN EXPONENTIALLY WEIGHTED OPTIMAL QUADRATURE FORMULA IN THE SPACE $W_2^{(1,0)}$ OF PERIODIC FUNCTIONS

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Abstract: This work studies the problem of construction of the optimal quadrature formula in the sense of Sard in the Hilbert space $W_2^{(1,0)}(0,1]$ of periodic, complex-valued functions for numerical calculation of Fourier integrals. Here a quadrature sum consists of a linear combination of the given function values on a uniform mesh. The optimal quadrature formula is obtained by minimizing the norm of the error functional with respect to coefficients.

Key words: The Hilbert space, the Fourier integrals, the Fourier coefficients, the error functional, periodic functions, the extremal function, optimal coefficients, optimal quadrature formula.

1. Introduction and statement of the problem

In this work using the Sobolev method [3] for the approximate calculation of the Fourier integrals

$$I[\varphi,\omega] = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx$$

the optimal quadrature formula is constructed and the square of the norm of the error functional for the constructed optimal quadrature formula is calculated.

Let $W_2^{(1,0)}(0,1]$ be a Hilbert space of periodic, complex-valued functions $\varphi(x), x\in(0,1]$ which is defined as

$$W_2^{(1,0)}(0,1]=\{\varphi:(0,1]\to \mathbb{C} | \varphi \text{ is abs.cont. and } \varphi' \in L_2(0,1]\}.$$

Furthermore, it should be noted that every element of the space $W_2^{(1,0)}(0,1]$ satisfies the following condition of 1-periodicity

$$\varphi(x+\beta)=\varphi(x) \ \ for \ \ x\in\mathbb{R}, \beta\in\mathbb{Z}.$$

The inner product for the functions φ and ψ in this space is defined as

$$\langle\varphi,\psi\rangle_{W_2^{(1,0)}}=\int_0^1(\varphi'(x)+\varphi(x))\left(\overline{\psi}'(x)+\overline{\psi}(x)\right)dx, \qquad (1)$$

where $\overline{\psi}$ is the complex conjugate to the function ψ .

2. Exponentially weighted optimal quadrature formula

We consider a quadrature formula of the following form

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \cong \sum_{k=1}^N C_k\varphi(hk), \qquad (2)$$

where $\omega\in\mathbb{Z}$, $\varphi\in W_2^{(1,0)}, C_k$ are the coefficients of the quadrature formula and $N\in\mathbb{N}, h=\frac{1}{N}$.

The error of the quadrature formula is given as follows

$$\ell(\varphi)=\int_0^1 e^{2\pi i\omega x}\varphi(x)dx-\sum_{k=1}^N C_k\varphi(hk), \qquad (3)$$

and

$$\ell(x)=e^{2\pi i\omega x}-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\quad(4)$$

is the periodic error functional of the quadrature formula (2), here δ is the Dirac delta-function.

The error (3) of the quadrature formula (2) is a linear functional in $W_2^{(1,0)*}$. The absolute value of the error (3) is estimated by the Cauchy-Schwarz inequality as follows

$$|\ell(\varphi)|\le\left\|\ell\,|\,W_2^{(1,0)*}\right\|\cdot\left\|\varphi\,|\,W_2^{(1,0)}\right\|,$$

where

$$\left\|\ell\,|\,W_2^{(1,0)*}\right\|=\sup_{\varphi,\left\|\varphi|W_2^{(1,0)}\right\|\ne0}\frac{|\ell(\varphi)|}{\|\varphi\|}\qquad(5)$$

is the norm of the error functional (4).

The problem of constructing an optimal quadrature formula for the approximate calculation of the integral is as follows.

Problem 2.1. Find the coefficients C_k that give the minimum value to the norm $\left\|\ell\,|\,W_2^{(1,0)*}\right\|$, and calculate the following quantity

$$\left\|\overset{o}{\ell}\,|\,W_2^{(1,0)*}\right\|=\inf_{C_k}\left\|\ell\,|\,W_2^{(1,0)*}\right\|.$$

We note that the coefficients C_k which are the solution for Problem 1 are called *the optimal coefficients* and the quadrature formula (2) with these coefficients is said to be *the optimal quadrature formula in the sense of Sard* [2].

3. Main Results

To calculate the norm (5), we use *the extremal function* ψ_ℓ for the error functional ℓ (see [3]) that satisfies the following equality:

$$\ell(\psi_\ell) = \left\| \ell \,|\, W_2^{(1,0)*} \right\| \cdot \left\| \psi_\ell \,|\, W_2^{(1,0)} \right\|. \quad (6)$$

Since $W_2^{(1,0)}$ is the Hilbert space by the Riesz representation theorem in a Hilbert space for the error functional ℓ and for any φ from $W_2^{(1,0)}$ there exists an element $\psi_\ell \in W_2^{(1,0)}$ that satisfies the equality

$$\ell(\varphi) = \langle \psi_\ell, \varphi \rangle_{W_2^{(1,0)}}, \qquad (7)$$

where $\langle \varphi, \psi_\ell \rangle_{W_2^{(1,0)}}$ is the inner product of the functions ψ_ℓ and φ defined by equality (1). In addition, the equality $\left\| \ell \,|\, W_2^{(1,0)*} \right\| = \left\| \psi_\ell \,|\, W_2^{(1,0)} \right\|$ is fulfilled. So, taking into account equality (6), we derive

$$\ell(\psi_\ell) = \left\| \ell \,|\, W_2^{(1,0)*} \right\|^2.$$

Integrating by parts the right-hand side of (7), keeping in mind periodicity of functions, for ψ_ℓ we have

$$\psi_\ell''(x) - \psi_\ell(x) = -\ell(x). \quad (8)$$

Further, we give the main results of this work.

Theorem 3.1 *The solution of equation (8) is the extremal function* ψ_ℓ *of the error functional* ℓ *and it has the following expression*

$$\psi_\ell(x) = \frac{e^{-2\pi i\omega x}}{4\pi^2\omega^2+1} + \sum_{k=1}^{N}\overline{C}_k \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta(x-hk)}}{4\pi^2\beta^2+1},$$

where $\overline{C}_k$ is the complex conjugate to the function C_k.

Theorem 3.2 *If $\varphi \in W_2^{(1,0)}$, then the following formulas are valid for the optimal coefficients of the quadrature formula (2) with the error functional (4)*

$$\overset{o}{C}_k = \frac{2}{4\pi^2\omega^2+1} \cdot \frac{e^{2h}+1-2e^h\cos(2\pi\omega h)}{e^{2h}-1} \cdot e^{2\pi i\omega hk}, \text{ for } k=1,2,\dots,N.$$

Theorem 3.3 *In the space $W_2^{(1,0)}$ for the norm of the error functional (4) of the optimal quadrature formula, the following holds*

$$\left\| \overset{o}{\ell} \,|\, W_2^{(1,0)*} \right\|^2 = \frac{1}{4\pi^2\omega^2+1} - \frac{2}{\left(4\pi^2\omega^2+1\right)^2} \cdot \frac{e^{2h}+1-2e^h\cos(2\pi\omega h)}{h\left(e^{2h}-1\right)}. \qquad (9)$$

Remark 3.1 *It should be noted that from (9) we obtain*

$$\left\| \overset{o}{\ell} \right\|^2 = \frac{1}{12}h^2 - \left(\frac{4\pi^2\omega^2-3}{360}\right) \cdot h^4 + O(h^6),$$

i.e., the order of convergence of the optimal quadrature formula of the form (2) is $O(h)$.

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