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ψ- Riman-Liuvill kasr tartibli integrallar uchun Hardi-Littlevud tengsizligi haqidagi ayrim teoremalarning isboti

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Annotasiya: Lebeg fazosida ψ -Riman-Liuvill tipidagi kasrli integralning chegaralanganligi va Hardi-Littlevud tengsizligi haqidagi teoremalari isbotlangan

Kalit so'zlar: Hardi va Littlevud tengsizligi, Riman-Liuvill integrali, ψ-Riman-Liuvill kasr integrali, Gyolder tengsizligi, musbat o'suvchi funksiya.

1928-yilda Hardi va Littlevud Riman-Liuvill integralining chegaralanganligini isbotlashdi. Klassik Hardi tengsizligi qism integrali uchun quyidagicha:

$$\left\| x^{\beta-\alpha} \int_0^x \frac{f(y)}{y^{\beta}(x-y)^{1-\alpha}} dy \right\|_{L^p(0,b)} \leq C \| f \|_{L^p(0,b)},$$

bunda $0 < \alpha < 1$, $\alpha - \frac{1}{p} < \beta < \frac{1}{q}$, $\frac{1}{q} + \frac{1}{p} = 1$ va $0 < b \leq \infty$.

Ushbu mavzuda ψ-Riman-Liuvill kasr integrallarining chegaralanganligini ko'rib chiqamiz.

1-teorema. Faraz qilaylik $\alpha>0, 1\le p\le\infty, -\infty<a<b<\infty$ va $\psi(x)$ musbat o‘suvchi funksiya bo‘lsin. U holda $L^p(a,b)$ fazoda $I_{a+}^{\alpha,\psi}$ operator chegaralangan va ushbu

$$\left\|I_{a+}^{\alpha;\psi}\varphi\right\|_{L^p(a,b)}\le\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\varphi\|_{L^p(a,b)} \quad \textbf{(1)}$$

tengsizlik o‘rinli bo‘ladi.

Isbot. Ataylik $p=\infty$ bo‘lsin, u holda ushbu

$$\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}dt\le\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}$$

tengsizlikka asosan, barcha $x\in(a,b]$ uchun quyidagini olamiz:

$$\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt\le\|\varphi\|_{L^\infty(a,b)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}dt\le$$

$$\le\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\|\varphi\|_{L^\infty(a,b)}.$$

p = 1 bo‘lsin, bundan esa quyidagiga ega bo‘lamiz

$$\int_a^b\left|I_{a+}^{\alpha;\Psi}\varphi(x)\right|dx=\int_a^b\left|\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt\right|dx\le$$

$$\le\frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dtdx$$

$$\le\frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x\psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dtdx$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^b|u(t)|\int_t^b\psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dxdt$$

$$=\frac{1}{\Gamma(\alpha+1)}\int_a^b(\psi(b)-\psi(t))^{\alpha}|\varphi(t)|\,dt\le\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\int_a^b|\varphi(t)|dt$$

Endi $1<p<\infty$ bo‘lgan holatni ko‘rib chiqamiz. $q>0$ va $\frac{1}{p}+\frac{1}{q}=1$ bo‘lsin.

Gyolder tengsizligiga asosan, quyidagiga ega bo‘lamiz

$$\left|I_{a+}^{\alpha;\psi}\varphi(x)\right| = \left|\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|dt\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{1}{q}}[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{1}{p}}|\varphi(t)|dt \leq$$

$$\leq \frac{1}{\Gamma(\alpha)}(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}dt)^{1/q}(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p dt)^{1/p} \leq$$

$$\leq \frac{1}{\Gamma(\alpha)}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{1/q}(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|u(t)|^p dt)^{1/p}.$$

Shunday qilib, $\psi'(x)$ o‘suvchi ekanligini hisobga olsak, quyidagini olamiz

$$\int_a^b \left|I_{a+}^{\alpha;\Psi}\varphi(x)\right|^p dx \leq \frac{1}{[\Gamma(\alpha)]^p}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{p/q}\int_a^b\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p dtdx \leq$$

$$\leq \frac{1}{[\Gamma(\alpha)]^p}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{p/q}\int_a^b\int_a^x \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p dtdx$$

$$= \frac{1}{[\Gamma(\alpha)]^p}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{p/q}\int_a^b|\varphi(t)|^p\int_t^b \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}dxdt$$

$$\leq \frac{1}{[\Gamma(\alpha)]^p}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{p/q}\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\int_a^b|\varphi(t)|^p dt$$

$$= \frac{1}{[\Gamma(\alpha)]^p}(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha})^{\frac{p}{q}+1}\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\int_a^b|\varphi(t)|^p dt.$$

Demak, oxirgi tengsizlikdan (1) kelib chiqadi. □

2-teorema. Faraz qilaylik $0<\alpha<1, -\infty<a<b<\infty$ va $\psi(x)$ musbat o‘suvchi funksiya bo‘lsin. Agar $1\leq p<\frac{1}{\alpha}$ va $1\leq q<\frac{p}{1-\alpha p}$, $\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}$ bo‘lsa, u holda $I_{a+}^{\alpha,\psi}$ operator $L^p(a,b)$ fazoni $L^q(a,b)$ fazoga ko‘chiradi va ushbu

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}u\right\|_{L^q(\alpha,b)} \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\left[\psi'(b)\right]^{r-1}(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\|u\|_{L^p(\alpha,b)}$$

tengsizlik o‘rinli bo‘ladi.

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Isboti. Endi p = 1 va $1 \le q < \dfrac{1}{1-\alpha}$ bo‘lgan holatini ko‘rib chiqamiz. q = 1 bo‘lgan hol 1-teoremada isbotlangan, shuning uchun $1 \le q < \dfrac{1}{1-\alpha}$ holni qaraymiz. Gyolder tengsizligiga asosan, quyidagiga ega bo‘lamiz

$$\left|I_{a+}^{\alpha;\psi}\varphi(x)\right| \le \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|\varphi(t)\right|dt$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|\varphi(t)\right|^{\frac{1}{q}}\left|\varphi(t)\right|^{1-\frac{1}{q}}dt$$

$$\le \frac{1}{\Gamma(\alpha)}(\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^q\left|\varphi(t)\right|dt)^{1/q}\left(\int_a^x\left|\varphi(t)\right|dt\right)^{1-\frac{1}{q}}.$$

Shundan so‘ng,

$$\left|\mathrm{I}_{a+}^{\alpha,\psi}\varphi(x)\right|^q \le \frac{1}{[\Gamma(\alpha)]^q}\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha}-1]^q\left|\varphi(t)\right|dt\left\|\varphi\right\|_{L^1(a,b)}^{q-1}$$

Bundan esa,

$$\left\|\mathrm{I}_{a+}^{\alpha,\psi}\varphi(x)\right\|_{L^p(a,b)}^q \le \frac{1}{[\Gamma(\alpha)]^q}\int_a^b\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}-1]^q\left|\varphi(t)\right|dtdx\left\|\varphi\right\|_{L^1(a,b)}^{q-1}$$

$$\le \frac{1}{[\Gamma(\alpha)]^q}\int_a^b\left|\varphi(t)\right|\int_t^b[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^q dxdt\left\|\varphi\right\|_{L^1(a,b)}^{q-1}$$

$$=\frac{1}{[\Gamma(\alpha)]^q}\int_a^b\left|\varphi(t)\right|[\psi'(b)]^{q-1}\int_t^b\psi'(x)(\psi(x)-\psi(t))^{(\alpha-1)q}dxdt\left\|\varphi\right\|_{L^1(a,b)}^{q-1}$$

$$=\frac{[\psi'(b)]^{q-1}}{[\Gamma(\alpha)]^q}\int_a^b\left|u(t)\right|\frac{(\psi(b)-\psi(t))^{1-(1-\alpha)q}}{1-(1-\alpha)q}dt\left\|\varphi\right\|_{L^1(a,b)}^{q-1}$$

$$\le \frac{[\psi'(b)]^{q-1}(\psi(b)-\psi(a))^{1-(1-\alpha)q}}{(1-(1-\alpha)q)[\Gamma(\alpha)]^q}\left\|\varphi\right\|_{L^1(a,b)}^q.$$

Demak, $\forall 1 \le q < \dfrac{1}{1-\alpha}$ uchun

$$\left|I_{a+}^{\alpha;\Psi}u\right|_{L^q_{(a,b)}} \le (\frac{[\psi'(b)]^{q-1}(\psi(b)-\psi(a))^{1-(1-\alpha)q}}{1-(1-\alpha)q[\Gamma(\alpha)]^q})^{1/q}\left\|\varphi\right\|_{L^1_{(a,b)}}$$ bo‘ladi.

Isbotni davom ettirib, $1<p<\frac{1}{\alpha}$ va $p\le q<\frac{p}{1-\alpha p}$ holni ko‘rib chiqamiz. $p=q$ hol 1-teoremada isbotlangan, shuning uchun biz faqat ishda $p<q<\frac{p}{1-\alpha p}$ holni qaraymiz. Quyidagi belgilashni kiritamiz $\delta=\frac{1}{p}-\frac{1}{q}$,

bunda 0 < δ < α < 1. r ni topamiz

$$\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}=1-\delta\Rightarrow r=\frac{1}{1-\delta}.$$

$r<q$ e’tiborga olsak, Gyolder tengsizligini qo‘llaymiz

$$\left|I_{a+}^{\alpha;\Psi}\varphi(x)\right|=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt=$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{r}{q}}|\varphi(t)|^{\frac{p}{q}}[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{1-\frac{r}{q}}|\varphi(t)|^{p(\frac{1}{p}-\frac{1}{q})}\,dt$$

$$\le\frac{1}{\Gamma(\alpha)}\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r|\varphi(t)|^p\,dt\right)^{1/q}\left(\int_a^x|\varphi(t)|^p\,dt\right)^{\frac{1}{p}-\frac{1}{q}}$$

$$\times\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r\,dt\right)^{1-\frac{1}{p}}=\frac{1}{\Gamma(\alpha)}\left(\frac{[\psi'(x)]^{r-1}(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}}$$

$$\times\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r|\varphi(t)|^p\,dt\right)^{1/q}\left(\int_a^x|\varphi(t)|^p\,dt\right)^{\frac{1}{p}-\frac{1}{q}}$$

Teorema shartiga asosan, oxirgi ifoda quyidagi shartlarda o‘rinli bo‘ladi:

$$\frac{1}{r}=1-\delta>1-\alpha\Rightarrow(1-\alpha)r<1.$$

Bu shartga ko‘ra, quyidagi tengsizlik o‘rinli bo‘ladi

$$\left\|I_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^q(a,b)}^q=\int_a^b\left|I_{\alpha+}^{\alpha;\psi}\varphi(x)\right|^q dx$$

$$\le\frac{1}{[\Gamma(\alpha)]^q}\int_a^b\Big[\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r|\varphi(t)|^p\,dt\,\|\varphi\|_{L^p(a,b)}^{q-p}$$

$$\left(\frac{[\psi'(x)]^{r-1}(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\Big]dx$$

$$\le \frac{\|\varphi\|_{L^p(a,b)}^{q-p}}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\int_a^b\int_a^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^r|\varphi(t)|^p\,dtdx$$

$$\le \frac{\|\varphi\|_{L^p(a,b)}^{q-p}}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\frac{[\psi'(b]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\|\varphi\|_{L^p(a,b)}^{q-p}$$

$$=\frac{1}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}C(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})+1}\|\varphi\|_{L^p(a,b)}^q$$

Demak, $\forall p\le q<\dfrac{p}{1-\alpha p}$ uchun

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^q(\alpha,b)}\le\frac{1}{\Gamma(\alpha)}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\|\varphi\|_{L^p(a,b)}\qquad \text{bo‘ladi.}$$

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