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SONLI QATORLARGA DOIR BA'ZI MISOLLAR TAHLILI

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Annotatsiya: Ushbu maqolada matematik analizda sonli qatorlarning yaqinlashuv xususiyatlari tahlil qilinadi. To'rtta turli misol orqali qatorlarning yaqinlashuvini tekshirishda qo'llaniladigan metodlar — taqqoslash orqali, qatorlarning yaqinlashuvi va absolyut yaqinlashuvi misollarda ko'rib chiqiladi. Maqolaning maqsadi — qachon va nima sababdan qatorlar absolyut yoki shartli yaqinlashishini yoki aksincha, yaqinlashmasligini chuqur tushunishga yordam berishdir. Ushbu ish oliy matematika bilan shug'ullanayotgan talabalar va o'qituvchilar uchun nazariy va amaliy qo'llanma bo'lib xizmat qiladi.

Kalit so'zlar: sonli qatorlar, yaqinlashuv, absolyut yaqinlashuv, shartli yaqinlashuv, Shtolts teoremasi.

Misol 1. α ning qanday qiymatlarida $\sum_{n=1}^{\infty} a_n$ qator yaqinlashishini aniqlang.

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Bizga berilgan:

$$a_n = \left(\sqrt{n+1}-\sqrt{n}\right)^{\alpha} \cdot \ln\frac{2n+1}{2n-1}$$

yechish:

Soddalashtiramiz $\left(\sqrt{n+1}-\sqrt{n}\right)$:

$$\sqrt{n+1}-\sqrt{n} = \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

$n \to \infty$, biz bilamizki

$$\sqrt{n+1}+\sqrt{n} \approx 2\sqrt{n}\,.$$

$$\sqrt{n+1}-\sqrt{n} \sim \frac{1}{2\sqrt{n}} = 0*\left(\frac{1}{\sqrt{n}}\right).$$

Soddalashtiraylik $\ln\frac{2n+1}{2n-1}$:

$$\ln\frac{2n+1}{2n-1} = \ln(1+\frac{2n+1}{2n-1}-1) = \ln(1+\frac{(2n+1)-(2n-1)}{2n-1})$$

x uchun biz bilamizki $\ln(1+x) \approx x$. Bu yerda $n \to \infty, \frac{2}{2n-1} \to 0.$

$$\ln\left(1+\frac{2}{2n-1}\right) \sim \frac{2}{2n-1} \sim \frac{2}{2n} = \frac{1}{n}.$$

$$\ln\frac{2n+1}{2n-1} = 0*\left(\frac{1}{n}\right).$$

Bu ifodani biz:

$$a_n \sim \left(\frac{1}{\sqrt{n}}\right)^{\alpha} \cdot \frac{1}{n} = \frac{1}{n^{\frac{\alpha}{2}}} \cdot \frac{1}{n} = \frac{1}{n^{\frac{\alpha}{2}+1}}.$$

$$b_n = \frac{1}{n^{1+\frac{\alpha}{2}}}.$$

Taqqoslashimizga ko’ra, agar $\lim \frac{a_n}{b_n}$ chekli musbat son bo’lsa, $\sum a_n$ ham yaqinlashuvchi $\sum b_n$ ham yaqinlashuvchi.

Bizga ma’lumki $p-$qaror $\sum_{n=1}^{\infty} \frac{1}{n^p}$ yaqinlashuvchi agar $p>1$.

Bizda $p=1+\frac{\alpha}{2}$.

$\sum a_n$ qatorga yaqinlashuvchi $1+\frac{\alpha}{2}>1$.

Bu ifodadan $\frac{\alpha}{2}>0$ ya’ni $\alpha>0$.

$\sum_{n=1}^{\infty} a_n$ qator ham $\alpha>0$ da yaqinlashadi.

Misol 2. Ushbu $\sum_{n=1}^{\infty} \frac{\cos n}{n^{\alpha}}$

Qator α ning qanday qiymatlarida

a) Absolyut yaqinlashuvchi

b) Shartli yaqinlashuvchi bo’lishini aniqlang.

Yechish: Birinchi navbatda berilgan qator α ning qanday qiymatlarida yaqinlashuvchi bo’lishini aniqlaymiz.Bunda Dirixle alomatidan foydalanamiz.

Agar $a_n = \frac{1}{n^\alpha}$ va $b_n = \cos n$

1) $\alpha > 0$ bo'lganda $\{a_n\}\downarrow$ va $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0,$

2) $B_n = \sum\limits_{k=1}^{n} b_k = \frac{\cos\frac{n+1}{2}\cdot\sin\frac{n}{2}}{\sin\frac{1}{2}}$ va $|B_n| \le \frac{1}{\sin\frac{1}{2}}$ bo'ladi.$\Rightarrow$ alomatiga ko'ra $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{\cos n}{n^\alpha}$ qator $\alpha > 0$ bo'lganda yaqinlashadi. $\alpha \le 0$ bo'lganda esa bu qator uzoqlashadi, chunki $\alpha \le 0$ bo'lganda qator yaqinlashishiga tekshiramiz. $|\frac{\cos n}{n^\alpha}| \le \frac{1}{n^\alpha}$ va $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ umumlashgan garmonik qatorning $\alpha > 1$ da yaqinlashuvchi bo'lishidan $\alpha > 1$ da $\sum\limits_{n=1}^{\infty} \frac{|\cos n|}{n^\alpha}$ qatorning yaqinlashishini hosil qilamiz.

Endi $0 < \alpha \le 1$ bo'lganda berilgan qatorning absolyut yaqinlashuvchi emasligini, ya'ni $\sum\limits_{n=1}^{\infty} \frac{|\cos n|}{n^\alpha}$ qatorning uzoqlashishini ko'rsatamiz.

$$\frac{|\cos n|}{n^\alpha} \ge \frac{\cos^2 n}{n^\alpha} = \frac{1+\cos 2n}{2n^\alpha} = \frac{1}{2n^\alpha} + \frac{\cos 2n}{2n^\alpha},$$

tengsizlik hamda $\sum\limits_{n=1}^{\infty} \frac{\cos 2n}{2n^\alpha}$ qatorning Dirixle alomatiga ko'ra yaqinlashuvchi bo'lishi va x_0 qatorning uzoqlashuvchi ekanligidan $\{\sum\limits_{n} S_n(x)\}$: qatorning ham uzoqlashuvchi ekanligini, taqqoslash alomatiga ko'ra $S_n(x)$ qatorning uzoqlashuvchiligini hosil qilamiz.

Shunday qilib, $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n^\alpha}$ qator

a) $\alpha > 1$ da absolyut yaqinlashuvchi,

b) $0 < \alpha \le 1$ da shartli yaqinlashuvchi bo'lar ekan.

Misol 3. Aytaylik $\{a_n\}_{n=0}^{\infty}$ sonli ketma-ketlik uchun $0 < a_0 < \pi$ va

$$a_n = \frac{1}{n}\sum_{k=0}^{n-1} arctg\left(a_k\right), \quad n \ge 1$$

bo'lsin. U holda $\{a_n\sqrt{\ln\left(n\right)}\}_{n=1}^{\infty}$ sonli ketma-ketlik yaqinlashuvchiligini ko'rsating va limni toping.

Yechish:

Ketma-ketlikdan:

$$a_{n+1} = \frac{\sum_{k=0}^{n-1} arctg\left(a_k\right) + arctg\left(a_n\right)}{n+1} = \frac{na_n + arctg\left(a_n\right)}{n+1}$$

Oraliqqa e'tibor bersak $0 < a_0 < \pi$ va $arctg\left(x\right) \le x$

$$0 < a_{n+1} < \frac{na_n + a_n}{n+1} = a_n$$

Bu ketma-ketlik $(a_n)_n$ musbat va kamayib boruvchi va chegaralangan $(n+1)l = nl + \sin(l)$ ifodani qanoatlantiradi. $l = 0$ dan tashqari Taylor qatoriga ko'ra

$$a_{n+1} = \frac{na_n + a_n - \dfrac{a_n^{\ 3}}{3} + 0\left(a_n^3\right)}{n+1} = a_n\left(1 - \frac{a_n^2}{3(n+1)} + \frac{0\left(a_n^2\right)}{n+1}\right)$$

va

$$\frac{1}{a_{n+1}^2} = \frac{1}{a_n^2}\left(1 - \frac{a_n^2}{3(n+1)} + \frac{0\left(a_n^2\right)}{n+1}\right)^{-2} = \frac{1}{a_n^2}\left(1 + \frac{2a_n^2}{3\left(n+1\right)} + \frac{0\left(a_n^2\right)}{n+1}\right) = \frac{1}{a_n^2} + \frac{2}{3\left(n+1\right)} + \frac{0\left(1\right)}{n+1}$$

Stolz-Cesaro teoremasiga asosan,

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$$\lim_{n\to\infty}\left(a_n\sqrt{\ln(n)}\right)^2=\lim_{n\to\infty}\frac{\ln(n)}{\frac{1}{a_n{}^2}}\Rightarrow\lim_{n\to\infty}\frac{\ln(n+1)-\ln(n)}{\frac{1}{a_{n+1}^2}-\frac{1}{a_n{}^2}}=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{2}{3(n+1)}+\frac{0(1)}{n+1}}=\frac{3}{2}$$

$a_n\sqrt{\ln(n)}\geq 0$ ifodadan bu xulosaga kelamiz $\lim\limits_{n\to\infty}a_n\sqrt{\ln(n)}=\sqrt{\frac{3}{2}}$.

Misol 4. Hisoblang.

$$\sum_{k=1}^{n}\tan^2\left(\frac{\pi}{2n+1}\right).$$

Yechish:

Ko'rib chiqaylik $P_n(x)=\left(x^2+1\right)^{n+\frac{1}{2}}\cdot\sin(2n-1)\arctan(x)$. $P_n(x)$ ko'phad.

Bunga e'tibor bersak :

$$2i\sin(n\alpha)=e^{\ln\alpha}-e^{-\ln\alpha}=\left(\cos(\alpha)+i\sin(\alpha)\right)^n-\left(\cos(\alpha)-i\sin(\alpha)\right)^n=$$

$$\sum_{k=0}^{n}\binom{n}{k}\cos^k\alpha\sin^{n-k}\alpha\left(i^{n-k}-(-i)^{n-k}\right)$$

$$\sum_{k=0}^{n}\binom{n}{k}\cos^k\alpha\sin^{n-k}\alpha\left(e^{\frac{\pi i(n-k)}{2}}-e^{\frac{\pi i(n-k)}{2}}\right)=2i\sin^n\alpha\sum_{k=0}^{n}\binom{n}{k}\cot^k\alpha\sin\left(\frac{\pi(n-k)}{2}\right)$$

$$P_n(x)=\left(x^2+1\right)^{n+1/2}\cdot\sin^{2n+1}(\arctan x)\sum_{k=0}^{2n+1}\binom{2n+1}{k}\cot^k(\arctan x)\sin\left(\frac{\pi(2n+1-k)}{2}\right)$$

$$=x^{2n+1}\sum_{k=0}^{2n+1}\binom{2n+1}{k}\frac{1}{x^k}\sin\left(\frac{\pi(2n+1-k)}{2}\right)$$

$$=(-1)^n x^{2n+1}+(-1)^{n-1}\binom{2n+1}{2}x^{2n-1}+\ldots+\binom{2n+1}{2n}x.$$

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Bilamizki $x_k = \frac{\tan \pi}{2n+1}, k = -n, \ldots, n.$ Aniq $x_{-n}, \ldots, x_n$, P_n ning ildizlari va $\deg P_n = 2n+1$, uning boshqa ildizlari yo'q.Vyeta teoremasi bo'yicha $\sum_{i=-n}^{n} x_i = 0$ va

$$\sum_{-n \le i \le n} x_i x_j = -\binom{2n+1}{2}.$$

FOYDALANILGAN ADABIYOTLAR:

1) А.Саьдуллаев,Х Мансуров Г Худойберганов А Ворисов Р Гуломов Matematik analiz kursidan misol va masalalar to'plami.

2) Б.П.Демитович Сборник задач и упреажнений по математичискомй анализу Маскуа "наука" 1990-год.

3) B.A.Shoimqulov.T.T.Tuychiyev.D.H.Djumaboyev.Matematik analiz mustaqil ishlari.

4) Open Mathematical Olympiad for University Students (OMOUS-2023).