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On some Gibbs measures of a coupled Ising-Ising model

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Abstract: In this paper, we study the translation-invariant Gibbs measures of a coupled Ising-Ising model defined on a Cayley tree of order $k \geq 2$. The model is derived as a special case of the coupled Ising-Potts model when the number of Potts states $q = 2$. We consider a Hamiltonian with spin configurations represented by $(s,\sigma) \in \{-1,1\} \times \{1,2\}$ and obtain a system of nonlinear equations characterizing the corresponding Gibbs measures. For the particular case $k = 2$, the system is analyzed in detail. We prove that if the interaction parameter satisfies the condition $\theta + \theta^{-1} > 2 + 2$, then the system admits at least three distinct translation-invariant Gibbs measures. This result highlights the occurrence of a phase transition in the model and contributes to the understanding of multi-component spin systems on hierarchical structures.

Keywords: Gibbs measures, coupled Ising-Ising model, Cayley tree, translation-invariance, Hamiltonian, phase transition, statistical mechanics.

Here we give a result related to Gibbs measures of a Hamiltonian defined on a Cayley tree $\Gamma^k = (V,L)$ (see [1] for main definitions). Namely, following [2] we consider a coupled Ising-Potts model on Cayley trees of order $k \geq 2$. This has spin vectors (s,σ), where $s \in I = \{-1,1\}$ and $\sigma \in \Phi = \{1,...,q\}$, $q \geq 2$.

A coupled configuration (s,σ) on V is then defined as a function $x \in V \; 7\!\longrightarrow (s(x),\sigma(x)) \in I\times\Phi$. The following Hamiltonian is called the coupled Ising-Potts model:

$$H(s,\sigma) = -J \sum_{<x,y>\in L} s(x)s(y)\delta_{\sigma(x)\sigma(y)} \quad (1)$$

where $J \in R$ is a constant, $<x,y>$ is nearest neighbor vertices and δ_{ij} is the Kroneker's symbol.

In [2] it is proven that to each Gibbs measure of Hamiltonian corresponds a vector-valued function $h_x = (h_{\epsilon,i,x})_{(\epsilon,i)\in I\times\Phi}$ defined on vertices of the Cayley tree that satisfies for any $x \in V$ the following equations:

$$z_{\epsilon,i,x} = \prod_{y\in S(x)} \frac{(1-\theta^{-\epsilon})(\theta^{\epsilon} z_{1,i,y} - z_{-1,i,y}) + \sum_{j=1}^{q-1}(z_{-1,j,y} + z_{1,j,y}) + z_{1,q,y} + 1}{\theta + \theta^{-1} z_{1,q,y} + \sum_{j=1}^{q-1}(z_{-1,j,y} + z_{1,j,y})}, \quad (2)$$

$$z_{1,q,x} = \prod_{y\in S(x)} \frac{(1-\theta_{-1})(\theta z_{1,q,y} - 1) + \sum_{j=1}^{q-1}(z_{-1,j,y} + z_{1,j,y}) + z_{1,q,y} + 1}{\theta + \theta^{-1} z_{1,q,y} + \sum_{j=1}^{q-1}(z_{-1,j,y} + z_{1,j,y})}, \quad (3)$$

where $\epsilon \in I,\ z_{\epsilon,i,x} = \exp\{h_{\epsilon,i,x} - h_{-1,q,x}\},\ i = 1,...,q-1,\ \theta = \exp(J\beta)$, and $S(x)$ is the set of direct successors of x.

We consider Gibbs measures which are translatsion-invariant, i.e correspond to solutions of the form:

$z_{\epsilon,i,x} \equiv z_{\epsilon,i}$, for all $x \in V$.

Here we consider the case $q = 2$, $k = 2$. Note that in case $q = 2$ the Potts model coincides with the Ising model. This is why we call our model as an Ising-Ising model. Then by (2), (3) denoting $u_1 = z_{-1,1}$ $v_1 = z_{1,1}$, $v_2 = z_{1,2}$ we get:

$$u_1 = \left(\frac{(1-\theta^{-1})(\theta u_1 - v_1) + u_1 + v_1 + v_2 + 1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2, \quad (5)$$

$$v_1 = \left(\frac{(1-\theta^{-1})(\theta v_1 - u_1) + u_1 + v_1 + v_2 + 1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2, \quad (6)$$

$$v_2 = \left(\frac{(1-\theta^{-1})(\theta v_2 - 1) + u_1 + v_1 + v_2 + 1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2. \qquad (7)$$

Through making equation simpler we get:

$$u_1 = \left(\frac{\theta^{-1} v_1 + \theta u_1 + v_2 + 1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2, \qquad (8)$$

$$v_1 = \left(\frac{\theta^{-1} u_1 + \theta v_1 + v_2 + 1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2, \qquad (9)$$

$$v_2 = \left(\frac{\theta^{-1} + \theta v_2 + u_1 + v_1}{\theta + \theta^{-1} v_2 + u_1 + v_1}\right)^2. \qquad (10)$$

Let $u_1 = u$, $v_1 = v$, and $v_2 = 1$. We have following system of equations:

$$\begin{cases} u = \left(\frac{\theta u + \theta^{-1} v + 2}{\theta + \theta^{-1} + u + v}\right)^2 \\ v = \left(\frac{\theta^{-1} u + \theta v + 2}{\theta + \theta^{-1} + u + v}\right)^2 \end{cases},$$

In the case when $u = v$, we have to solve the following equation from the system.

$$u = \left(\frac{\theta^{-1} u + \theta u + 2}{\theta + \theta^{-1} + 2u}\right)^2 \qquad (11)$$

Let's denote $\theta + \theta^{-1}$ as a and take the square root of both sides of the equation. As a result, we will obtain the following equation:

$$\sqrt{u} = \frac{au + 2}{a + 2u} \qquad (12)$$

Let $\sqrt{u} = x$ and we have following equation

$$2x^3 + ax - ax^2 - 2 = 0 \qquad 13)$$

$$2(x^3 - 1) + a(x - x^2) = 0 \qquad 14)$$

$$(x - 1)(2x^2 + 2x + 1 - ax) = 0$$

15)

As a result we get three $x_1 = 1,\ x_2 = \frac{a-2-\sqrt{a^2-4a-4}}{4}$ solutions

$x_3 = \frac{a-2+\sqrt{a^2-4a-4}}{4}$ and

Considering that $u = x^2$ analyzing the expression under the square root is enough. $a > 2+2$, i.e $\theta + \theta^{-1} > 2+2$, system has three solutions: (1,1,1)

$$\left(\left(\frac{a-2-\sqrt{a^2-4a-4}}{4}\right)^2, \left(\frac{a-2-\sqrt{a^2-4a-4}}{4}\right)^2, 1\right)$$ and
$$\left(\left(\frac{a-2+\sqrt{a^2-4a-4}}{4}\right)^2, \left(\frac{a-2+\sqrt{a^2-4a-4}}{4}\right)^2, 1\right)$$
.

As mentioned above to each solution of the system (2), (3) corresponds a Gibbs measure.

Therefore we have the following result:

Theorem 1. For the coupled Ising-Ising model, if $\theta + \theta^{-1} > 2+\sqrt{2}$ then there are at least three translation-invariant Gibbs measures.

References:

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